

Majorization bounds for distribution function

Ismihan Bairamov

Department of Mathematics, Izmir University of Economics
Izmir, Turkey. E-mail: ismihan.bayramoglu@ieu.edu.tr

September 2, 2011

Abstract

Let X be a random variable with distribution function F , and X_1, X_2, \dots, X_n are independent copies of X . Consider the order statistics $X_{i:n}$, $i = 1, 2, \dots, n$ and denote $F_{i:n}(x) = P\{X_{i:n} \leq x\}$. Using majorization theory we write upper and lower bounds for F expressed in terms of mixtures of distribution functions of order statistics, i.e. $\sum_{i=1}^n p_i F_{i:n}$ and $\sum_{i=1}^n p_i F_{n-i+1:n}$. It is shown that these bounds converge to F for a particular sequence $(p_1(m), p_2(m), \dots, p_n(m)), m = 1, 2, \dots$ as $m \rightarrow \infty$.

1 Introduction

Let X_1, X_2, \dots, X_n be independent and identically distributed (iid) random variables with distribution function (cdf) F and $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ be the corresponding order statistics. Order statistics are very important in the theory of statistics and its applications. The theory of order statistics is well documented in David (1981), David and Nagaraja (2003), Arnold et al. (1992) and in many research papers dealing with different theoretical properties and applications of ordinary order statistics and general models of ordered random variables. Order statistics play a special role in statistical theory of reliability, since they can be interpreted as the failure times of n units with lifetimes X_1, X_2, \dots, X_n placed on a life test. A system of n components is called a k -out-of- n system if it functions if and only if at least k components function and therefore, life time of such a system is $X_{n-k+1:n}$. (see Barlow and Proschan, 1975). Consider a coherent system composed of n identical components with lifetimes X_1, X_2, \dots, X_n having distribution function F . Then the distribution function of the system lifetime T can be expressed as a convex combination of order statistics $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ using Samanige signatures (p_1, p_2, \dots, p_n) as follows:

$$P\{T \leq x\} = \sum_{i=1}^n z_i F_{i:n}(x), \quad (1)$$

where $F_{i:n}(x) = P\{X_{i:n} \leq x\}$ and $z_i = P\{T = X_{i:n}\}$, $i = 1, 2, \dots, n$ are signatures (see Samaniego 2007). The system reliability can be expressed as

$$P\{T > x\} = \sum_{i=1}^n z_i \bar{F}_{i:n}(x),$$

where $\bar{F}_{i:n}(x) = 1 - F_{i:n}(x)$. If $z_i = 1/n$, $i = 1, 2, \dots, n$, then $P\{T \leq x\} = \frac{1}{n} \sum_{i=1}^n F_{i:n}(x) = F(x)$. This means that if the system signature vector is $(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, then the system reliability is the same with the reliability of a single component. For a general coherent system, assuming that the system reliability is known, is it possible to determine the reliability of the components? The results presented in this paper allow to answer partially this question, i.e. it follows that for a particular choice of signatures, \bar{F} can be approximated by the reliability of the system.

In general, in this note we consider mixtures of distribution functions of order statistics $K_n(x) := \sum_{i=1}^n p_i F_{i:n}(x)$ and $H_n(x) := \sum_{i=1}^n p_i F_{n-i+1:n}(x)$ and using well known inequalities of majorization theory we show that for a particular choice of p_i 's, $H_n(x) \leq F(x) \leq K_n(x)$ for all $x \in \mathbb{R}$. It is shown that the similar inequalities can be written for the sample mean and mixtures of order statistics. For a particular choice of vector (p_1, p_2, \dots, p_n) the L_2 distance between $H_n(x)$ and $K_n(x)$ can be made as small as we want.

2 Main Results

Let $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ and $a_{[1]} \geq a_{[2]} \geq \dots \geq a_{[n]}$ denote the components of \mathbf{a} in decreasing order. The vector \mathbf{a} is said to be majorized by the vector \mathbf{b} and denoted by $\mathbf{a} \prec \mathbf{b}$, if

$$\sum_{i=1}^k a_{[i]} \leq \sum_{i=1}^k b_{[i]} \text{ for } k = 1, 2, \dots, n-1$$

and

$$\sum_{i=1}^n a_{[i]} = \sum_{i=1}^n b_{[i]}.$$

The details of the theory of majorization can be found in Marshall et al. (2011). The following two theorems are important for our study.

Proposition 1 Denote $D = \{(x_1, x_2, \dots, x_n) : x_1 \geq x_2 \geq \dots \geq x_n\}$, $\mathbf{a} = (a_1, a_2, \dots, a_n)$, $\mathbf{b} = (b_1, b_2, \dots, b_n)$. The inequality

$$\sum_{i=1}^n a_i x_i \leq \sum_{i=1}^n b_i x_i$$

holds for all $(x_1, x_2, \dots, x_n) \in D$ if and only if $\mathbf{a} \prec \mathbf{b}$ in D . (Marshall et al. 2011, page 160).

Proposition 2 *The inequality*

$$\sum_{i=1}^n a_i x_i \leq \sum_{i=1}^n b_i x_i$$

holds whenever $x_1 \leq x_2 \leq \dots \leq x_n$ if and only if

$$\begin{aligned} \sum_{i=1}^k a_i &\geq \sum_{i=1}^k b_i, \quad k = 1, 2, \dots, n-1 \\ \sum_{i=1}^n a_i &= \sum_{i=1}^n b_i. \end{aligned}$$

(Marshall et al. 2011, page 639).

Now, let X_1, X_2, \dots, X_n be iid random variables with cdf F , and survival function $\bar{F} = 1 - F$. Let $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ be corresponding order statistics and $F_{i:n}(x) = P\{X_{i:n} \leq x\}$. We are interested in mixtures $\sum_{i=1}^n p_i \bar{F}_{i:n}(x)$ of cdf's of order statistics, where $p_i \geq 0, p_1 \geq p_2 \geq \dots \geq p_n$ and $\sum_{i=1}^n p_i = 1$.

Denote

$$D_+^1 = \{(x_1, x_2, \dots, x_n) : x_i \geq 0, i = 1, 2, \dots, n; x_1 \geq x_2 \geq \dots \geq x_n, \sum_{i=1}^n x_i = 1\}.$$

Lemma 1 *Let $(p_1, p_2, \dots, p_n) \in D_+^1$. Then*

$$H_n(x) \equiv \sum_{i=1}^n p_i F_{n-i+1:n}(x) \leq F(x) \leq \sum_{i=1}^n p_i F_{i:n}(x) \equiv K_n(x) \text{ for all } x \in \mathbb{R} \quad (2)$$

and the equality holds if and only if $(p_1, p_2, \dots, p_n) = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$.

Proof. Since $F_{1:n}(x) \geq F_{2:n}(x) \geq \dots \geq F_{n:n}(x)$ for all $x \in \mathbb{R}$, and $(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}) \prec (p_1, p_2, \dots, p_n)$, the right hand side of the inequality (2) follows from Proposition 1 and left hand side follows from Proposition 2. ■

Corollary 1 *Let $\mathbf{p} = (p_1, p_2, \dots, p_n) \in D_+^1$, $\mathbf{q} = (q_1, q_2, \dots, q_n) \in D_+^1$ and $\mathbf{p} \prec \mathbf{q}$. Then*

$$\sum_{i=1}^n q_i F_{n-i+1:n}(x) \leq \sum_{i=1}^n p_i F_{n-i+1:n}(x) \leq F(x) \leq \sum_{i=1}^n p_i F_{i:n}(x) \leq \sum_{i=1}^n q_i F_{i:n}(x)$$

Example 1 Let $F(x) = x$, $0 \leq x \leq 1$. Then $H_n(x) = \sum_{i=1}^n q_i \sum_{k=i}^n \binom{n}{k} x^k (1-x)^{n-k}$ and $K_n(x) = \sum_{i=1}^n q_i \sum_{k=n-i+1}^n \binom{n}{k} x^k (1-x)^{n-k}$. Let $n = 3$ and $\mathbf{q} = (q_1, q_2, q_3) = (\frac{5}{9}, \frac{3}{9}, \frac{1}{9})$. By simple calculations we have

$$H_3(x) = \frac{1}{3}(2x^2 + x) \leq F(x) \leq \frac{1}{3}(5x - 2x^2) = K_3(x).$$

Let $\mathbf{p} = (p_1, p_2, p_3) = (\frac{6}{15}, \frac{5}{15}, \frac{4}{15})$, then the functions $H_3(x)$ and $K_3(x)$ for this \mathbf{p} are as follows:

$$H_3(x) = \frac{1}{5}(x^2 + x) \leq F(x) \leq \frac{6}{5}(x - x^2) = K_3(x).$$

It is clear that $\mathbf{p} \prec \mathbf{q}$. Below we present the graphs of the functions $H_3(x)$, $F(x) = x$, and $K_3(x)$ for two different vectors \mathbf{q} and \mathbf{p} :

Figure 1. $H_3(x)$, $F(x) = x$, and $K_3(x)$,

$$\text{for } (q_1, q_2, q_3) = (\frac{5}{9}, \frac{3}{9}, \frac{1}{9}) \text{ and } (p_1, p_2, p_3) = (\frac{6}{15}, \frac{5}{15}, \frac{4}{15}).$$

For $(q_1, q_2, q_3) = (\frac{5}{9}, \frac{3}{9}, \frac{1}{9})$, the L_2 distance between the functions $H_3(x)$, $K_3(x)$ can be calculated and it is $d(H_3(x), K_3(x)) = \int_0^1 (H_3(x) - K_3(x))^2 dx = \frac{8}{135} \simeq 0.059259$. For $(p_1, p_2, p_3) = (\frac{6}{15}, \frac{5}{15}, \frac{4}{15})$ the distance is $\int_0^1 (H_3(x) - K_3(x))^2 dx = \frac{2}{375} \simeq 0.005333$.

Note that the vector $(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ is the "smallest" in the sense of majorization, among the vectors $(p_1, p_2, \dots, p_n) \in D_+^1$, i.e. $(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}) \prec (p_1, p_2, \dots, p_n)$ for all $(p_1, p_2, \dots, p_n) \in D_+^1$. Now the problem of interest is: for a given n , how small can the distance between $H_n(x)$ and $K_n(x)$ be made by appropriate choice of the vector (p_1, p_2, \dots, p_n) ?

The following theorem answers this question.

Theorem 2 There exists a sequence $\mathbf{p}(m) = (p_1(m), p_2(m), \dots, p_n(m)) \in D_+^1$, $m = 1, 2, \dots$ such that

$$H_n^{(m)}(x) \equiv \sum_{i=1}^n p_i(m) F_{n-i+1:n}(x) \leq F(x) \leq \sum_{i=1}^n p_i(m) F_{i:n}(x) \equiv K_n^{(m)}(x) \text{ for all } x \in \mathbb{R} \quad (3)$$

and

$$\lim_{m \rightarrow \infty} \sum_{i=1}^n p_i(m) F_{n-i+1:n}(x) = \lim_{m \rightarrow \infty} \sum_{i=1}^n p_i(m) F_{i:n}(x) = F(x) \text{ for all } x \in \mathbb{R}. \quad (4)$$

Furthermore,

$$\int_{-\infty}^{\infty} \left| K_n^{(m)}(x) - H_n^{(m)}(x) \right| dx = o\left(\frac{1}{m^{1-\alpha}}\right), \quad 0 < \alpha < 1. \quad (5)$$

Proof. Consider $p_i(m) = \frac{m+n-i+1}{a_n(m)}$, $i = 1, 2, \dots, n$; $m \in \{0, 1, 2, \dots\}$, where

$a_n(m) = nm + \frac{n(n+1)}{2}$. It is clear that $p_1(m) \geq p_2(m) \geq \dots \geq p_n(m)$ and $\sum_{i=1}^n p_i(m) = 1$. Since $(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}) \prec (p_1(m), p_2(m), \dots, p_n(m))$ then from Lemma 1 we have

$$\sum_{i=1}^n p_i(m) F_{n-i+1:n}(x) \leq F(x) \leq \sum_{i=1}^n p_i(m) F_{i:n}(x). \quad (6)$$

Since

$$\lim_{m \rightarrow \infty} p_i(m) = \lim_{m \rightarrow \infty} \frac{m+i}{nm + \frac{n(n+1)}{2}} = \frac{1}{n}, \quad i = 1, 2, \dots, n,$$

and (4) follows. To prove (5) consider the L_1 distance between $K_n^{(m)}(x)$ and $H_n^{(m)}(x)$. We have

$$\begin{aligned} \Delta_m &\equiv \int_{-\infty}^{\infty} \left| K_n^{(m)}(x) - H_n^{(m)}(x) \right| dx \\ &= \int_{-\infty}^{\infty} \left| \sum_{i=1}^n p_i(m) F_{i:n}(x) - \sum_{i=1}^n p_i(m) F_{n-i+1:n}(x) \right| dx \\ &= \int_{-\infty}^{\infty} \left| \sum_{i=1}^n p_i(m) F_{i:n}(x) - F(x) + F(x) - \sum_{i=1}^n p_i(m) F_{n-i+1:n}(x) \right| dx \\ &= \int_{-\infty}^{\infty} \left| \sum_{i=1}^n \left(p_i(m) - \frac{1}{n} \right) F_{i:n}(x) + \left(\frac{1}{n} - p_i(m) \right) F_{n-i+1:n}(x) \right| dx \\ &\leq \sum_{i=1}^n \left| p_i(m) - \frac{1}{n} \right| \int_{-\infty}^{\infty} |F_{i:n}(x) - F_{n-i+1:n}(x)| dx \\ &\leq (p_1(m) - \frac{1}{n}) c_n = \frac{\frac{1}{m} \frac{n(n+1)}{2}}{n^2 + \frac{n^2(n+1)}{2} \frac{1}{m}} c_n, \end{aligned}$$

where $c_n = \sum_{i=1}^n \int_{-\infty}^{\infty} |F_{i:n}(x) - F_{n-i+1:n}(x)| dx$. ■

In Figure 3 the graphs of $H_3(x)$, $K_3(x)$ for $n = 3$ in case of standard normal distribution $N_{0,1}(x)$ for a vector $(p_1, p_2, p_3) = (2/3, 2/9, 1/9)$, which clearly is not a member of the sequence $\mathbf{p}(m)$. The numerical calculations in Maple 13 show that $\int_{-\infty}^{\infty} |K_n^{(m)}(x) - H_n^{(m)}(x)| dx = 0.30903$.

Figure 2. Graphs of $H_3(x) \leq N_{0,1}(x) \leq K_3(x)$,
 $n = 3$ and $(p_1, p_2, p_3) = (2/3, 2/9, 1/9)$

The members of the sequence $\mathbf{p}(m)$, $m = 1, 2, \dots$ are most "uniform", and according to the basic idea of majorization they must allow better approximation than any other vector. To illustrate the rate of convergence in case of standard normal distribution we present in Figure 3 below, the graphs of $K_n^{(m)}(x) \leq N_{0,1}(x) \leq H_n^{(m)}(x)$. Some numerical values of Δ_m for different values of m are presented in Table 1.

Figure 3. Graphs of $K_n^{(m)}(x) \leq N_{0,1}(x) \leq H_n^{(m)}(x)$, $n = 3$
 $m = 2, 3, 10$

Table 1. Values of Δ_m

m	1	2	3	4	5
Δ_m	0.34337	0.13735	0.10301	0.08241	0.06867
m	10	15	20	25	30
Δ_m	0.03434	0.02423	0.018729	0.01526	0.01288

Remark 1 It is clear that using Proposition 1 and 2 and using order statistics $X_{i:n}$, instead of $F_{i:n}$ we have similar to Lemma 1 and Theorem 1 results for order statistics. Let $\bar{X} = \sum_{i=1}^n X_i$. Then, for a sequence $\mathbf{p}(m) =$

$(p_1(m), p_2(m), \dots, p_n(m)) \in D_+^1$, $m = 1, 2, \dots$ it is true that

$$X_m^L \equiv \sum_{i=1}^n p_i(m) X_{i:n} \leq \bar{X} \leq \sum_{i=1}^n p_i(m) X_{n-i+1:n} = X_m^U \text{ a.s.} \quad (7)$$

and

$$\lim_{m \rightarrow \infty} \left[\sum_{i=1}^n p_i(m) X_{n-i+1:n} - \sum_{i=1}^n p_i(m) X_{i:n} \right] = 0 \text{ a.s..} \quad (8)$$

Furthermore,

$$E |X_m^U - X_m^L| = o\left(\frac{1}{m^{1+\alpha}}\right), \quad 0 < \alpha < 1.$$

From (8) it follows that

$$\sum_{i=1}^n p_i(m) \mu_{i:n} \leq E(X) \leq \sum_{i=1}^n p_i(m) \mu_{n-i+1:n}, \quad (9)$$

where $\mu_{i:n} = E(X_{i:n})$. The rate of convergence can be estimated as

$$\begin{aligned} \delta_m &= \left| \sum_{i=1}^n p_i(m) \mu_{n-i+1:n} - \sum_{i=1}^n p_i(m) \mu_{i:n} \right| \\ &\leq \frac{\frac{1}{m} \frac{n(n+1)}{2}}{n^2 + \frac{n^2(n+1)}{2} \frac{1}{m}} C_n, \end{aligned}$$

$$\text{where } C_n = \sum_{i=1}^n |\mu_{n-i+1:n} - \mu_{i:n}|.$$

References

- [1] Arnold, B., Balakrishnan, N. and Nagaraja, H.N. (1992) *A First Course in Order Statistics*. John Wiley & Sons, Inc.
- [2] David, H. (1981) *Order Statistics*. Second Edition, Wiley, New York.
- [3] David, H.A. and Nagaraja, H.N. (2003) *Order Statistics*, Third Edition, Wiley, New York.
- [4] Barlow, R.E. and Proschan, F. (1975) *Statistical Theory of Reliability and Life Testing*. Holt, Rinehart and Winston, Inc.
- [5] Marshal, A. W. , Olkin, I. and Arnold, B.C. (2011) *Inequalities: Theory of Majorization and Its Applications*. Second edition. Springer.
- [6] Samaniego, F.J. (2007) *System Signatures and their Applications in Engineering Reliability*. Springer.